

Global Solutions of Evolutionary Faddeev Model With Small Initial Data

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Abstract

We consider the Cauchy problem for evolutionary Faddeev model corresponding to maps from the Minkowski space \mathbb{R}^{1+n} to the unit sphere \mathbb{S}^2 , which obey a system of non-linear wave equations. The nonlinearity enjoys the null structure and contains semi-linear terms, quasi-linear terms and unknowns themselves. We prove that the Cauchy problem is globally well-posed for sufficiently small initial data in Sobolev space.

Keywords: Faddeev model, global existence, quasi-linear wave equations, semi-linear wave equations.

1 Introduction

Denote an arbitrary point in $(n+1)$ -dimensional Minkowski space $\mathbb{R} \times \mathbb{R}^n$ by $z = (t, x) = (x^\alpha)_{0 \leq \alpha \leq n}$, the space-time derivatives of a function by

$$\partial = (\partial_t, \nabla) = (\partial_\alpha)_{0 \leq \alpha \leq n}.$$

We raise and lower indices with the Minkowski metric $\eta = (\eta_{\alpha\beta}) = \eta^{-1} = (\eta^{\alpha\beta}) = \text{diag}(1, -1, -1, -1)$.

To describe the Faddeev model, let us consider Sobolev mappings from the Minkowski space $(\mathbb{R} \times \mathbb{R}^n, \eta)$, $n \geq 2$ to the unit sphere \mathbb{S}^2 :

$$\mathbf{n} : (\mathbb{R} \times \mathbb{R}^n, \eta) \rightarrow \mathbb{S}^2 \tag{1.1}$$

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and the Lagrangian density governing the evolution of the fields \mathbf{n} (see Faddeev [6, 7, 8]):

$$\mathcal{L}(\mathbf{n}) = \frac{1}{2} \partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n} - \frac{1}{4} (\partial_\mu \mathbf{n} \wedge \partial_\nu \mathbf{n}) (\partial^\mu \mathbf{n} \wedge \partial^\nu \mathbf{n}).$$

Then solutions of the Faddeev model can be characterized variationally as critical points of the action integral

$$\mathcal{A}(\mathbf{n}) = \int_{\mathbb{R} \times \mathbb{R}^n} \mathcal{L}(\mathbf{n}) dx dt. \quad (1.2)$$

The equations of motion of the Faddeev model takes the form,

$$\mathbf{n} \wedge \partial_\mu \partial^\mu \mathbf{n} + [\partial_\mu (\mathbf{n} \cdot [\partial^\mu \mathbf{n} \wedge \partial^\nu \mathbf{n}])] \partial_\nu \mathbf{n} = 0, \quad (1.3)$$

which is the Euler-Lagrange equation of $\mathcal{L}(\mathbf{n})$ in local coordinates (see Faddeev [6, 7, 8] and Lin-Yang [21] and references therein).

The Faddeev model (1.3) was introduced to model elementary particles by using continuously extended, topologically characterized, relativistically invariant, locally concentrated, soliton-like fields. The model is not only important in the area of quantum field theory but also provides many interesting and challenging mathematical problems, see for examples [26], [27], [28], [25], [23], [5], [2] and [24]. There have been a lot of interests in recent years in studying mathematical issues of static Faddeev model (see Lin-Yang [17, 18, 19, 20] and review papers by Faddeev [8] and Lin-Yang [21]). However, the corresponding evolutionary equations (1.3), which turn out to be unusual quasi-linear wave equations, are still untouched to our best knowledge (see also Lin-Yang [21]).

In the case of $n \geq 3$, there are now classical and well developed theories on global well-posedness for quasi-linear wave equations with null structure and small initial data, see for examples, Christodoulou and Klainerman [4], Lindblad-Rodnianski [22], Sideris [29]. Such theories can be easily employed to solve the evolutionary system of the Faddeev model also when $n \geq 3$. The aim of this paper is to prove the global well-posedness of the Cauchy problem of the Faddeev model (1.3) in \mathbb{R}^{1+2} under the assumption that the initial data is small in some generalized Sobolev space. These results provide a starting point for further studies of evolutions of interacting particle like approximate solutions, see [26] and [24].

Theorem 1.1. *Suppose that $n_{10}, n_{20}, n_{11}, n_{21} \in C_0^\infty(\mathbb{R}^2)$ with $s \geq 9$ and*

$$\|n_{10}\|_{H^{s+2}}, \quad \|n_{20}\|_{H^{s+2}}, \quad \|n_{11}\|_{H^{s+1}}, \quad \|n_{21}\|_{H^{s+1}} \leq \epsilon, \quad n_{30} = \sqrt{1 - n_{10}^2 - n_{20}^2}.$$

Then there exists a small positive constant ϵ_0 such that the Faddeev model (1.3) with the initial data

$$n_i(0, x) = n_{i0}(x), \quad \partial_t n_j(0, x) = n_{j1}(x), \quad 1 \leq i \leq 3, \quad 1 \leq j \leq 2$$

is well-posed globally in time provided that $\epsilon \leq \epsilon_0$.

The nonlinearity in Faddeev model (1.3) (see also system (3.1) in section 3 and system (4.1) in section 4) enjoys the so-called null structure, which can be used to explore better decay estimates of solutions, see [3, 12, 29, 1]. In the two space dimension, it seems the best result was due to Alinhac [1], where the author introduced the so-called "ghost weights" in the energies and proved a global existence result for a class of quasi-linear wave equations (without terms which are semi-linear and involving unknowns themselves) with small initial values and null conditions. For quasi-linear wave equations whose nonlinearities are cubic, and involve only the derivatives of unknowns, we refer the reader to Li Tatsien [16] or Hoshiga [9]. One notices, however, that the nonlinearity in the Faddeev model (1.3) contains both semi-linear and quasi-linear terms where the semi-linear terms are cubic and involve the unknowns themselves (see (3.1) in section 3). Technically, it becomes much more complicated since the estimates for unknowns themselves can not be obtained by the usual Klainerman's generalized energy estimates. We also find that Alinhac's method is difficult to apply to such kind of nonlinearities in the two space dimension.

To prove Theorem 1.1, We will need the following *a priori* estimates:

Theorem 1.2. *Let $s \geq 9$ and (n_1, n_2) be a global classical solution to (1.3) with initial data n_0 which is given in Theorem 1.3. Then there holds*

$$\begin{cases} \sum_{i=1}^2 \|\partial^i \mathbf{n}(t, \cdot)\|_{\Gamma, s, L^2} \leq M\epsilon(1+t)^\delta, \\ \|(n_1(t, \cdot), n_2(t, \cdot))\|_{\Gamma, s, L^2} \leq M\epsilon(1+t)^{\frac{1}{2}+2\delta}, \\ \|(n_1(t, \cdot), n_2(t, \cdot))\|_{\Gamma, s-2, L^\infty} \leq M\epsilon(1+t)^{-\frac{1}{2}}, \end{cases} \quad (1.4)$$

for some appropriately small positive constant δ and some positive constant M provided that the initial data satisfies

$$\|n_{10}\|_{H^{s+2}}, \quad \|n_{20}\|_{H^{s+2}}, \quad \|n_{11}\|_{H^{s+1}}, \quad \|n_{21}\|_{H^{s+1}} \leq \epsilon \quad (1.5)$$

for sufficiently small positive constant ϵ .

To show the energy estimates of unknowns themselves (see the second inequality in (1.4)), we shall use the following refined norms as in [15]:

$$\|f\|_{L^{p,q}} = \left(\int_0^\infty \|f(r\xi)\|_{L^q(S^{n-1})}^p r^{n-1} dr \right)^{\frac{1}{p}}.$$

Using this norm, we are able to get essentially optimal L^2 estimates for unknowns themselves:

$$\begin{aligned} \|u(t, \cdot)\|_{L^2} &\leq \|u(0, \cdot)\|_{L^2} + C(1+t)^{\frac{1}{2}} \left\{ \|\partial_t u(0, \cdot)\|_{L^{\frac{4}{3}}} \right. \\ &\quad \left. + \int_0^t \left(\|\square u(\tau, \cdot)\|_{L^{\frac{4}{3}, \chi_1}} + (1+\tau)^{-\frac{1}{2}} \|\square u(\tau, \cdot)\|_{L^{1,2}, \chi_2} \right) d\tau \right\}, \end{aligned}$$

where χ_1 is the characteristic function of $\{x : |x| \leq 1 + \frac{\tau}{2}\}$ and $\chi_2 = 1 - \chi_1$. The proof of the above estimate is presented in Theorem 3.1. See (2.1) and (2.2) for the definitions of the norms appearing on the right hand side of the above inequality. The crucial point in this *a priori* estimate is that it allows us to take the advantage of the faster time decay of u in the region of $\text{supp}\chi_1$ and extra time decay of u in the complement of $\text{supp}\chi_1$ which is usually due to the null structure of nonlinearities. The above L^2 estimate combining with the best $L^1 - L^\infty$ estimate (see Theorem 2.3 and Theorem 2.4) by Klainerman [14] and Hörmander [10] allows us to be able to get the *a priori* estimate in (1.4). We also point out that our Lemma 4.1 is not covered in the Lemma 4.1 of [1].

The analysis in this paper can be used to deal with nonlinear wave equations with semi-linear terms, quasi-linear terms involving unknowns themselves as well. The method can likely be also adopted to study the sharp lifespan of nonlinear wave equations $\square u = F(u, \partial u, \partial^2 u)$ in two and three space dimensions with both semi-linear terms and quasi-linear terms that may contain unknowns themselves.

The paper is organized as follows: In section 2, we review some basic estimates for solutions of linear wave equations and the notion of null forms. We prove then the L^2 estimate for solutions of the linear wave equations. The second and the third *a priori* estimates in the Theorem 1.2 are established in section 3. In the final section 4 we prove the first *a priori* estimate in Theorem 1.2.

2 Preliminaries and Estimates for The Linear Wave Equations

After some preliminary discussions, we shall prove certain energy estimates for solutions of the linear wave equations which are essential for establishing the inequalities in Theorem 1.2. We first introduce several notations:

$$\begin{cases} \|f\|_{L^{p,q}} = \left(\int_0^\infty \|f(r\xi)\|_{L^q(S^{n-1})}^p r^{n-1} dr \right)^{\frac{1}{p}}, \\ \|f\|_{L^\infty,q} = \sup_{r \geq 0} \|f(r\xi)\|_{L^q(S^{n-1})}. \end{cases} \quad (2.1)$$

It is easy to see that

$$\|f\|_{L^{p,p}} = \|f\|_{L^p}.$$

For any integer $s \geq 0$, real numbers $1 \leq p, q \leq \infty$ and any characteristic function $\psi(t, x)$, we will denote

$$\begin{cases} \|u(t, \cdot)\|_{\Gamma, s, L^{p,q}, \psi} = \sum_{|k| \leq s} \|\psi(t, \cdot) \Gamma^k u(t, \cdot)\|_{L^{p,q}}, \\ \|u(t, \cdot)\|_{\Gamma, s, L^{p,q}} = \sum_{|k| \leq s} \|\Gamma^k u(t, \cdot)\|_{L^{p,q}}. \end{cases} \quad (2.2)$$

Here as in Klainerman [14], we use the following vector fields(operators):

$$\Gamma = (\partial, L, \Omega), \quad (2.3)$$

where

$$\begin{cases} \partial = (\partial_\alpha)_{0 \leq \alpha \leq n} = (\partial_t, \nabla), \\ \Omega = (\Omega_{ij})_{1 \leq i, j \leq n, i \neq j}, \quad \Omega_{ij} = x_i \partial_j - x_j \partial_i, \\ L_0 = t \partial_t + r \partial_r = x_\alpha \partial_\alpha, \quad L_i = t \partial_i + x_i \partial_t. \end{cases}$$

Denote the wave operator by $\square = \partial_t^2 - \Delta$ and the Poisson product by $[\cdot, \cdot]$.

First of all, it is easy to check that the following Proposition holds:

Proposition 2.1. *For any multi-index α , we have*

$$[\square, \Gamma^\alpha] = \sum_{|\beta| \leq |\alpha| - 1} C_{\alpha\beta} \Gamma^\beta \square, \quad [\partial, \Gamma^\alpha] = \sum_{|\beta| \leq |\alpha| - 1} C'_{\alpha\beta} \Gamma^\beta \partial \quad (2.4)$$

for some constants $C_{\alpha\beta}$ and $C'_{\alpha\beta}$.

Concerning Klainerman's vector fields in (2.3), one also has following Proposition:

Proposition 2.2. *There exists a positive constant C such that*

$$|\partial^s u(t, x)| \leq C(1 + |t - |x||)^{-s} \sum_{|\alpha| \leq s} |\Gamma^\alpha u(t, x)| \quad (2.5)$$

holds for all smooth function $u(t, x)$.

Proof. In fact, (2.5) is obvious if $|t - |x|| \leq 1$. Otherwise, (2.5) follows from the following expressions:

$$\partial_t = \frac{tL_0 - x_i L_i}{t^2 - |x|^2}, \quad \partial_{x_j} = \frac{tL_j - x_j L_0 - x_k \Omega_{kj}}{t^2 - |x|^2}.$$

□

Next let us recall the $L^\infty - L^1$ estimate for linear wave equations, whose proof can be found in Klainerman [13].

Theorem 2.3. *Assume that u solves the Cauchy problem of the homogeneous linear wave equation in $\mathbb{R} \times \mathbb{R}^n$:*

$$\square u = 0, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x). \quad (2.6)$$

Then we have

$$\|u(t, \cdot)\|_{L^\infty} \leq \frac{C(\|u_0\|_{W^{n,1}} + \|u_1\|_{W^{n-1,1}})}{(1+t)^{\frac{n-1}{2}}} \quad (2.7)$$

for all $t \geq 0$.

The following estimate can be found in Hörmander [10].

Theorem 2.4. *Let u solve the Cauchy problem of the inhomogeneous linear wave equation in $\mathbb{R} \times \mathbb{R}^2$:*

$$\square u = f, \quad u(0, x) = u_t(0, x) = 0. \quad (2.8)$$

Then we have

$$|u(t, x)| \leq \frac{C \int_0^t \|f(\tau, \cdot)\|_{\Gamma, 1, L^1} (1 + \tau)^{-(\frac{1}{2}-l)} d\tau}{(1 + t + |x|)^{\frac{1}{2}} (1 + |t - |x||)^l}. \quad (2.9)$$

Here $0 \leq l \leq \frac{1}{2}$.

We will need some Sobolev type inequalities. The first one is the well-known Sobolev Imbedding theorem on the unit sphere \mathbb{S}^{n-1} centered at the origin:

Theorem 2.5. *Let $x = r\xi$, $r = |x|$. Then there holds*

$$\begin{cases} sp > n - 1 : |v(x)| = |v(r\xi)| \leq C \sum_{|k| \leq s} \|\Omega^k v(r\xi)\|_{L_\xi^p}, \\ sp < n - 1 : \|v(r\xi)\|_{L_\xi^q} \leq C \sum_{|k| \leq s} \|\Omega^k v(r\xi)\|_{L_\xi^p}, \\ \frac{1}{q} = \frac{1}{p} - \frac{s}{n-1}, \\ sp = n - 1 : \|v(r\xi)\|_{L_\xi^q} \leq C \sum_{|k| \leq s} \|\Omega^k v(r\xi)\|_{L_\xi^p}, \quad p \leq q < \infty \end{cases} \quad (2.10)$$

for all smooth function $v(x)$.

The second one is the Sobolev Imbedding theorem in a ball \mathbf{B}_λ with radius λ centered at the origin:

Theorem 2.6. *Let $\lambda > 0$. Then there exists a positive constant C independent of λ such that*

$$\begin{cases} sp > n : \|u\|_{L^\infty(B_\lambda)} \leq C \lambda^{-\frac{n}{p}} \sum_{|\alpha| \leq s} \lambda^{|\alpha|} \|\nabla^\alpha u\|_{L^p(B_\lambda)}, \\ sp < n : \|u\|_{L^q(B_\lambda)} \leq C \lambda^{-n(\frac{1}{p}-\frac{1}{q})} \sum_{|k| \leq s} \lambda^{|\alpha|} \|\nabla^\alpha u\|_{L^p(B_\lambda)}, \\ \quad q = \frac{np}{(n-sp)}, \frac{1}{q} = \frac{1}{p} - \frac{s}{n}, \\ sp = n : \|u\|_{L^q(B_\lambda)} \leq C \lambda^{-n(\frac{1}{p}-\frac{1}{q})} \sum_{|k| \leq s} \lambda^{|\alpha|} \|\nabla^\alpha u\|_{L^p(B_\lambda)}, \\ \quad p \leq q < \infty. \end{cases} \quad (2.11)$$

Proof. For $\lambda = 1$, these are standard Sobolev imbedding inequalities. When $\lambda \neq 1$, it follows from a simple scaling technique. \square

Let us improve (2.11) to get decaying type inequalities for smooth function $u(t, x)$ using the norms defined in (2.1) and (2.2).

Lemma 2.7. *Let χ_1 be the characteristic function of $\{x \mid |x| \leq 1 + \frac{t}{2}\}$. Then*

$$\left\{ \begin{array}{l} sp > n : \|u(t, \cdot)\|_{L^\infty, \chi_1} \leq C(1+t)^{-\frac{n}{p}} \|u(t, \cdot)\|_{\Gamma, s, L^p, \chi_1} \\ sp < n : \|u(t, \cdot)\|_{L^q, \chi_1} \leq C(1+t)^{-n(\frac{1}{p}-\frac{1}{q})} \|u(t, \cdot)\|_{\Gamma, s, L^p, \chi_1} \\ \quad \frac{1}{q} = \frac{1}{p} - \frac{s}{n}, \\ sp = n : \|u(t, \cdot)\|_{L^q, \chi_1} \leq C(1+t)^{-n(\frac{1}{p}-\frac{1}{q})} \|u(t, \cdot)\|_{\Gamma, s, L^p, \chi_1} \\ \quad p \leq q < \infty. \end{array} \right. \quad (2.12)$$

Proof. Letting $\lambda = \frac{t}{2} + 1$ in Theorem 2.6, and then using Proposition 2.2, one can easily check the above decaying type inequalities. \square

The following Lemma involves the estimate of Sobolev norms for composite functions, which can be easily proved by chain rules and Hölder inequality.

Lemma 2.8. *Let α be a non-negative integer and F be a smooth function with $F(w) = O(|w|^{1+\alpha})$ for $|w| \leq 1$. For any integer $s \geq 0$ and any characteristic function χ , there exists a positive constant C such that*

$$\left\{ \begin{array}{l} \alpha = 0 : \|F(w(t, \cdot))\|_{\Gamma, s, L^{p,q}, \chi} \leq C \|w(t, \cdot)\|_{\Gamma, s, L^{p,q}, \chi}, \\ \alpha \geq 1 : \|F(w(t, \cdot))\|_{\Gamma, s, L^{p,q}, \chi} \leq C \prod_{i=1}^{\alpha} \|w(t, \cdot)\|_{\Gamma, s, L^{p_i, q_i}, \chi} \|w(t, \cdot)\|_{\Gamma, s, L^{p_0, q_0}, \chi}, \\ \quad \frac{1}{p} = \sum_{i=0}^{\alpha} \frac{1}{p_i}, \quad \frac{1}{q} = \sum_{i=0}^{\alpha} \frac{1}{q_i} \end{array} \right. \quad (2.13)$$

holds for all w with $\|w(t, \cdot)\|_{\Gamma, [\frac{s}{2}], L^\infty} \leq 1$.

Finally, let us recall the definition of null structure satisfied by nonlinearity in nonlinear wave equations. For $0 \leq \alpha, \beta \leq n$, let

$$Q_{\alpha\beta}(f, g) = \partial_\alpha f \partial_\beta g - \partial_\alpha f \partial_\beta g, \quad Q(f, g) = \partial_t f \partial_t g - (\nabla f)(\nabla g).$$

$Q_{\alpha\beta}(f, g)$ and $Q(f, g)$ are called nonlinearities with null structure. Concerning the nonlinearities with null structure, we have

Lemma 2.9. *Let $Q_{\alpha\beta}(f, g)$ and $Q(f, g)$ are nonlinearities with null structure. Then one has*

$$|Q_{\alpha\beta}(f, g)(t, x)| + |Q(f, g)(t, x)| \leq \frac{C(|\Gamma f| |Dg| + |Df| |\Gamma g|)}{1+t}. \quad (2.14)$$

Proof. In fact, one can check the identities

$$\left\{ \begin{array}{l} Q_{ij}(f, g) = \frac{-\partial_t f \Omega_{ij} g + L_i f \partial_j g - L_j f \partial_i g}{t}, \\ Q_{0j}(f, g) = \frac{\partial_t f L_j g - L_j f \partial_t g}{t}, \\ Q(f, g) = \frac{\partial_t f L_0 g - \sum_{i=1}^2 L_i f \partial_i g}{t}, \end{array} \right.$$

which give (2.14) if t is large. In the case that t is small, (2.14) is obvious. \square

It is easy to check the following commutating property (see Klainerman [12]):

Lemma 2.10. *Let Γ be any vector field defined in (2.3), $Q_{\alpha\beta}(f, g)$ and $Q(f, g)$ be the nonlinearities with null structure as in Lemma 2.9. Then there holds*

$$\begin{cases} [\Gamma, Q_{\alpha\beta}] = \lambda^{\alpha\beta\gamma\delta} Q_{\gamma\delta}, \\ [\Gamma, Q] = \lambda Q_{\gamma\delta}, \end{cases}$$

where λ 's are constants and $[\Gamma, Q_{\alpha\beta}](f, g) = \Gamma Q_{\alpha\beta}(f, g) - Q_{\alpha\beta}(\Gamma f, g) - Q_{\alpha\beta}(f, \Gamma g)$.

3 Proof of Theorem 1.1

Let us first rewrite the Faddeev model (1.3) under geodesic normal coordinates (n_1, n_2) :

$$\begin{aligned} & \partial_\mu \partial^\mu \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} + \frac{\partial_\mu n_1 \partial^\mu n_1 + \partial_\mu n_2 \partial^\mu n_2}{1 - n_1^2 - n_2^2} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \\ & - \frac{n_2^2 \partial_\mu n_1 \partial^\mu n_1 + n_1^2 \partial_\mu n_2 \partial^\mu n_2 - 2n_1 n_2 \partial_\mu n_1 \partial^\mu n_2}{1 - n_1^2 - n_2^2} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \\ & + \frac{\partial_\mu \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right)}{\sqrt{1 - n_1^2 - n_2^2}} \begin{pmatrix} (1 - n_1^2) \partial_\nu n_2 + n_1 n_2 \partial_\nu n_1 \\ -(1 - n_2^2) \partial_\nu n_1 - n_1 n_2 \partial_\nu n_2 \end{pmatrix} = 0, \end{aligned} \quad (3.1)$$

which turns out to be quasi-linear wave equations. The local existence of classical solutions for quasilinear wave equations is well-known provided that the initial data belongs to Sobolev space $H^{s+2} \times H^{s+1}$ with $s \geq 1$ (see [11]). Consequently, our main Theorem 1.1 is just a corollary of the *a priori* estimates (1.4) in Theorem 1.2.

This section and section 4 are devoted to establishing the *a priori* estimates in Theorem 1.2. Our strategy is to use the continuity argument in the time variable t . By [11] and the assumptions on the initial data in (1.5), it is obvious that (1.4) is true for sufficiently small time t and some big constant M depending only on the norms of the initial data in Theorem 1.1. Let us assume that $T > 0$ is the biggest time such that (1.4) is true on $0 \leq t \leq T$. If $T = \infty$, then we are done. If $T < \infty$, we are going to prove that

$$\begin{cases} \sum_{i=1}^2 \|(\partial^i n_1(t, \cdot), \partial^i n_2(t, \cdot))\|_{\Gamma, s, L^2} < M\epsilon(1+t)^\delta, \\ \| (n_1(t, \cdot), n_2(t, \cdot)) \|_{\Gamma, s, L^2} < M\epsilon(1+t)^{\frac{1}{2}+2\delta}, \\ \| (n_1(t, \cdot), n_2(t, \cdot)) \|_{\Gamma, s-2, L^\infty} < M\epsilon(1+t)^{-\frac{1}{2}}, \end{cases} \quad (3.2)$$

for $0 \leq t \leq T$. By [11] again, we conclude that (1.4) is valid at least for $0 \leq t \leq T + \delta_0$ with a sufficiently small $\delta_0 > 0$, and hence we obtain a contradiction to the maximality of T . Thus (1.4) is valid for all time $t \geq 0$.

Consequently, our goal is to prove that (3.2) is true for $0 \leq t \leq T$ under the assumption that (1.4) is true for $0 \leq t \leq T < \infty$. Before doing that, let us prove the following Lemma concerning the following L^2 estimate of the unknown itself for linear wave equation:

Theorem 3.1. *Assume that u solve the linear wave equation in $\mathbb{R} \times \mathbb{R}^2$:*

$$\square u = f, \quad u(0, x) = u_0(x), u_t(0, x) = u_1(x). \quad (3.3)$$

Then we have

$$\begin{aligned} \|u(t, \cdot)\|_{L^2} &\leq \|u_0(\cdot)\|_{L^2} + C(1+t)^{\frac{1}{2}} \left\{ \|u_1(\cdot)\|_{L^{\frac{4}{3}}} \right. \\ &\quad \left. + \int_0^t \left(\|f(\tau, \cdot)\|_{L^{\frac{4}{3}, \chi_1}} + (1+\tau)^{-\frac{1}{2}} \|f(\tau, \cdot)\|_{L^{1,2}, \chi_2} \right) d\tau \right\}, \end{aligned} \quad (3.4)$$

where χ_1 is the characteristic function of $\{x : |x| \leq 1 + \frac{t}{2}\}$ and $\chi_2 = 1 - \chi_1$.

Proof. To prove (3.4), we compute that

$$\begin{aligned} \|u(t, \cdot)\|_{L^2} &\leq \|u_0(\cdot)\|_{L^2} + \left\| \frac{\sin(|\xi|t)}{|\xi|} \widehat{u_1}(\xi) \right\|_{L^2} \\ &\quad + \int_0^t \left\| \frac{\sin(|\xi|(t-\tau))}{|\xi|} \widehat{\chi_1 f(\tau, \cdot)}(\xi) \right\| d\tau + \int_0^t \left\| \frac{\sin(|\xi|(t-\tau))}{|\xi|} \widehat{\chi_2 f(\tau, \cdot)}(\xi) \right\| d\tau. \end{aligned}$$

Next we do the following straightforward computation

$$\begin{aligned} \left\| \frac{\sin(|\xi|t)}{|\xi|} \widehat{u_1}(\xi) \right\|_{L^2} &= \left\| \frac{\sin(|\eta|)}{|\eta|} \widehat{u_1}\left(\frac{\eta}{t}\right) \right\|_{L^2} \\ &\leq C \left\| (1+|\eta|)^{-1} \widehat{u_1}\left(\frac{\eta}{t}\right) \right\|_{L^2} = Ct^2 \left\| (1+|\eta|)^{-1} \widehat{u_1(tx)}(\eta) \right\|_{L^2} \\ &\leq Ct^2 \|u_1(tx)\|_{H^{-1}} \leq Ct^2 \|u_1(tx)\|_{L^{\frac{4}{3}}} = Ct^{\frac{1}{2}} \|u_1\|_{L^{\frac{4}{3}}}. \end{aligned}$$

A similar estimate also holds for $\int_0^t \left\| \frac{\sin(|\xi|(t-\tau))}{|\xi|} \widehat{\chi_1 f(\tau, \cdot)}(\xi) \right\| d\tau$.

Finally we compute

$$\begin{aligned}
& \left\| \frac{\sin(|\xi|(t-\tau))}{|\xi|} \widehat{\chi_2 f(\tau, \xi)} \right\| \\
& \leq C(t-\tau)^2 \|\chi_2 f(\tau, (t-\tau)x)\|_{H^{-1}} \\
& = C(t-\tau)^2 \sup_{v \in H^1} \frac{\int \chi_2 f(\tau, (t-\tau)x) v(x) dx}{\|v\|_{H^1}} \\
& \leq C(t-\tau)^2 \sup_{v \in H^1} \frac{\|\chi_2 f(\tau, (t-\tau)x)\|_{L^{1,2}} \|v\|_{L^{\infty,2}, |(t-\tau)y| \geq \frac{1+\tau}{2}}}{\|v\|_{H^1}} \\
& \leq C(t-\tau)^2 \|\chi_2 f(\tau, (t-\tau)x)\|_{L^{1,2}} \sup_{v \in H^1} \frac{\left(\sup_{r \geq \frac{1+\tau}{2(t-\tau)}} \int |v(r\xi)|^2 d\xi \right)^{\frac{1}{2}}}{\|v\|_{H^1}} \\
& \leq C \|f(\tau, x)\|_{L^{1,2}, \chi_2} \sup_{v \in H^1} \frac{\left(\sup_{r \geq \frac{1+\tau}{2(t-\tau)}} - \int_r^\infty \partial_r \int |v(r\xi)|^2 d\xi dr \right)^{\frac{1}{2}}}{\|v\|_{H^1}} \\
& \leq C \|f(\tau, x)\|_{L^{1,2}, \chi_2} \sqrt{\frac{t-\tau}{1+\tau}} \leq C \sqrt{\frac{1+t}{1+\tau}} \|f(\tau, x)\|_{L^{1,2}, \chi_2}.
\end{aligned}$$

The proof of Theorem 3.1 is thus completed. \square

Remark 3.2. It is easy to see that the following estimate

$$\|u(t, \cdot)\|_{L^2} \leq \|u_0(\cdot)\|_{L^2} + C(1+t)^{\frac{1}{2}} \left\{ \|u_1(\cdot)\|_{L^{\frac{4}{3}}} + \int_0^t \|f(\tau, \cdot)\|_{L^{\frac{4}{3}}} d\tau \right\} \quad (3.5)$$

is also true by using the same proof as that for Theorem 3.1. In fact, one deduces (3.5) in the case that $f(t, x)$ decays sufficiently fast outside the light cone $\{(t, x) : |x| \leq 1 + \frac{t}{2}\}$.

Now let us move on to show (3.2) for $0 \leq t \leq T$ under the assumption that (1.4) is true for $0 \leq t \leq T < \infty$. We shall first prove the second and third *a priori* estimates in (3.2), while we will prove the first inequality of (3.2) in section 4.

First of all, noting that $[\frac{s}{2}] + 5 \leq s$ for $s \geq 9$, one can easily deduce from Sobolev inequality and the third inequality in (1.4) that

$$\begin{aligned}
& \sum_{i=0}^2 \left\| (\partial^i n_1(t, \cdot), \partial^i n_2(t, \cdot)) \right\|_{\Gamma, [\frac{s}{2}] + 1, L^\infty} \\
& \leq C \left\| (n_1(t, \cdot), n_2(t, \cdot)) \right\|_{\Gamma, [\frac{s}{2}] + 3, L^\infty} \leq C \left\| (n_1(t, \cdot), n_2(t, \cdot)) \right\|_{\Gamma, s-2, L^\infty} \\
& \leq CM\epsilon(1+t)^{-\frac{1}{2}}.
\end{aligned} \quad (3.6)$$

Inequality (3.6) will be used repeatedly in the rest of this section and in section 4.

Estimates for $\|(n_1(t, \cdot), n_2(t, \cdot))\|_{\Gamma, s-2, L^\infty}$ in (3.2)

Let

$$\begin{aligned}
f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &= -\frac{\partial_\mu n_1 \partial^\mu n_1 + \partial_\mu n_2 \partial^\mu n_2}{1 - n_1^2 - n_2^2} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \\
&+ \frac{n_2^2 \partial_\mu n_1 \partial^\mu n_1 + n_1^2 \partial_\mu n_2 \partial^\mu n_2 - 2n_1 n_2 \partial_\mu n_1 \partial^\mu n_2}{1 - n_1^2 - n_2^2} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \\
&- \frac{\partial_\mu \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right)}{\sqrt{1 - n_1^2 - n_2^2}} \begin{pmatrix} (1 - n_1^2) \partial_\nu n_2 + n_1 n_2 \partial_\nu n_1 \\ -(1 - n_2^2) \partial_\nu n_1 - n_1 n_2 \partial_\nu n_2 \end{pmatrix}.
\end{aligned} \tag{3.7}$$

By Proposition 2.1, one has

$$\begin{cases} \square \Gamma^\alpha n_1 = \sum_{\beta \leq \alpha} C_\beta \Gamma^\beta f_1, \\ \square \Gamma^\alpha n_2 = \sum_{\beta \leq \alpha} C_\beta \Gamma^\beta f_2. \end{cases} \tag{3.8}$$

Consequently, by Theorem 2.3 and Theorem 2.4, we have

$$\begin{aligned}
\|n_1(t, \cdot)\|_{\Gamma, s-2, L^\infty} &\leq C \sum_{|\alpha| \leq s-2} \|\Gamma^\alpha n_1(t, \cdot)\|_{L^\infty} \\
&\leq C(1+t)^{-\frac{1}{2}} \sum_{|\alpha| \leq s-2} \left\{ \|\Gamma^\alpha n_1(0, \cdot)\|_{W^{2,1}} + \|\partial_t \Gamma^\alpha n_1(0, \cdot)\|_{W^{1,1}} \right. \\
&\quad \left. + \int_0^t [\|\Gamma^\alpha f_1(\tau, \cdot)\|_{W^{1,1}} (1+\tau)^{-\frac{1}{2}}] d\tau \right\} \\
&\leq C(1+t)^{-\frac{1}{2}} \left\{ \|(n_{10}, n_{20})\|_{H^{s+2}} + \|(n_{11}, n_{21})\|_{H^{s+1}} \right. \\
&\quad \left. + \int_0^t [\|f_1(\tau, \cdot)\|_{\Gamma, s-1, L^1} (1+\tau)^{-\frac{1}{2}}] d\tau \right\}.
\end{aligned} \tag{3.9}$$

Here we point out that one can use equations (3.7) to express $\Gamma^\alpha n_1$, $\partial_t(\Gamma^\alpha n_1)$, $\Gamma^\alpha n_2$ and $\partial_t(\Gamma^\alpha n_2)$ at time $t = 0$ in terms of the spacial derivatives of n_{10} , n_{20} , n_{11} and n_{21} . As a consequence, one has that

$$\begin{aligned}
&\sum_{|\alpha| \leq s-2} \left\{ \|\Gamma^\alpha n_1(0, \cdot)\|_{W^{2,1}} + \|\partial_t \Gamma^\alpha n_1(0, \cdot)\|_{W^{1,1}} \right\} \\
&\leq C(\|(n_{10}, n_{20})\|_{H^{s+2}} + \|(n_{11}, n_{21})\|_{H^{s+1}}).
\end{aligned}$$

Repeating the above argument, one also has

$$\begin{aligned}
\|n_2(t, \cdot)\|_{\Gamma, s-2, L^\infty} &\leq C(1+t)^{-\frac{1}{2}} \left\{ \|(n_{10}, n_{20})\|_{H^{s+2}} + \|(n_{11}, n_{21})\|_{H^{s+1}} \right. \\
&\quad \left. + \int_0^t [\|f_2(\tau, \cdot)\|_{\Gamma, s-1, L^1} (1+\tau)^{-\frac{1}{2}}] d\tau \right\}.
\end{aligned} \tag{3.10}$$

To proceed further, we need estimate $\|f_1(\tau, \cdot)\|_{\Gamma, s-1, L^1}$ and $\|f_2(\tau, \cdot)\|_{\Gamma, s-1, L^1}$.

First of all, Hölder inequality gives

$$\begin{aligned}
& \left\| \frac{n_1 (\partial_\mu n_1 \partial^\mu n_1 + \partial_\mu n_2 \partial^\mu n_2)}{1 - n_1^2 - n_2^2} \right\|_{\Gamma, s-1, L^1} \\
& \leq C \left\| \frac{n_1}{1 - n_1^2 - n_2^2} \right\|_{\Gamma, s-1, L^2} \left\| \partial_\mu n_1 \partial^\mu n_1 + \partial_\mu n_2 \partial^\mu n_2 \right\|_{\Gamma, [\frac{s-1}{2}], L^2} \\
& \quad + C \left\| \frac{n_1}{1 - n_1^2 - n_2^2} \right\|_{\Gamma, [\frac{s-1}{2}], L^\infty} \left\| \partial_\mu n_1 \partial^\mu n_1 + \partial_\mu n_2 \partial^\mu n_2 \right\|_{\Gamma, s-1, L^1}.
\end{aligned} \tag{3.11}$$

By (1.4) and Lemma 2.8, we have

$$\left\| \frac{n_1}{1 - n_1^2 - n_2^2} \right\|_{\Gamma, s-1, L^2} \leq C \|(n_1, n_2)\|_{\Gamma, s-1, L^2} \leq CM\epsilon(1 + \tau)^{\frac{1}{2} + 2\delta}.$$

By Lemma 2.9 and Lemma 2.10, we estimate

$$\begin{aligned}
& \left\| \partial_\mu n_1 \partial^\mu n_1 + \partial_\mu n_2 \partial^\mu n_2 \right\|_{\Gamma, [\frac{s-1}{2}], L^2} \\
& \leq C \sum_{|\alpha+\beta| \leq [\frac{s-1}{2}]} \left\| \partial_\mu \Gamma^\alpha n_1 \partial^\mu \Gamma^\beta n_1 + \partial_\mu \Gamma^\alpha n_2 \partial^\mu \Gamma^\beta n_2 \right\|_{L^2} \\
& \leq C(1 + \tau)^{-1} \sum_{|\alpha+\beta| \leq [\frac{s-1}{2}]} \left\| \partial \Gamma^\alpha n_1 \Gamma^\beta n_1 + \partial \Gamma^\alpha n_2 \Gamma^\beta n_2 \right\|_{L^2} \\
& \leq C(1 + \tau)^{-1} \|(\partial n_1, \partial n_2)\|_{\Gamma, s-3, L^2} \|(n_1, n_2)\|_{\Gamma, s-2, L^\infty} \\
& \leq C(M\epsilon)^2(1 + \tau)^{-\frac{3}{2} + \delta}.
\end{aligned}$$

Consequently, the above two estimates yield

$$\left\| \frac{n_1}{1 - n_1^2 - n_2^2} \right\|_{\Gamma, s-1, L^2} \left\| \partial_\mu n_1 \partial^\mu n_1 + \partial_\mu n_2 \partial^\mu n_2 \right\|_{\Gamma, [\frac{s-1}{2}], L^2} \leq C(M\epsilon)^3(1 + \tau)^{-1+3\delta}. \tag{3.12}$$

Similarly, using (1.4) and Lemma 2.8 one more time, we can deduce that

$$\left\| \frac{n_1}{1 - n_1^2 - n_2^2} \right\|_{\Gamma, [\frac{s-1}{2}], L^\infty} \leq CM\epsilon(1 + \tau)^{-\frac{1}{2}}.$$

By (1.4), Lemma 2.9 and Lemma 2.10, one thus has

$$\begin{aligned}
& \left\| \partial_\mu n_1 \partial^\mu n_1 + \partial_\mu n_2 \partial^\mu n_2 \right\|_{\Gamma, s-1, L^1} \\
& \leq C(1 + \tau)^{-1} \|(\partial n_1, \partial n_2)\|_{\Gamma, s-1, L^2} \|(n_1, n_2)\|_{\Gamma, s, L^2} \\
& \leq C(M\epsilon)^2(1 + \tau)^{-\frac{1}{2} + 3\delta}.
\end{aligned}$$

Consequently, one obtains that

$$\begin{aligned}
& \left\| \frac{n_1}{1 - n_1^2 - n_2^2} \right\|_{\Gamma, [\frac{s-1}{2}], L^\infty} \left\| \partial_\mu n_1 \partial^\mu n_1 + \partial_\mu n_2 \partial^\mu n_2 \right\|_{\Gamma, s-1, L^1} \\
& \leq C(M\epsilon)^3(1 + \tau)^{-1+3\delta}.
\end{aligned} \tag{3.13}$$

Combining (3.12), (3.13) with (3.11), we thus conclude

$$\left\| \frac{n_1(\partial_\mu n_1 \partial^\mu n_1 + \partial_\mu n_2 \partial^\mu n_2)}{1 - n_1^2 - n_2^2} \right\|_{\Gamma, s-1, L^1} \leq C(M\epsilon)^3(1 + \tau)^{-1+3\delta}. \quad (3.14)$$

A similar argument gives also that

$$\begin{aligned} & \left\| \frac{n_1(n_2^2 \partial_\mu n_1 \partial^\mu n_1 + n_1^2 \partial_\mu n_2 \partial^\mu n_2 - 2n_1 n_2 \partial_\mu n_1 \partial^\mu n_2)}{1 - n_1^2 - n_2^2} \right\|_{\Gamma, s-1, L^1} \\ & \leq C(M\epsilon)^5(1 + \tau)^{-2+3\delta}. \end{aligned} \quad (3.15)$$

To finish the estimate for $\|f_1(\tau, \cdot)\|_{\Gamma, s-1, L^1}$, it remains to bound

$$\left\| \frac{(1 - n_1^2) \partial_\nu n_2 + n_1 n_2 \partial_\nu n_1}{\sqrt{1 - n_1^2 - n_2^2}} \partial_\mu \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \right\|_{\Gamma, s-1, L^1}.$$

Using Hölder inequality, we can estimate the above quantity as follows:

$$\begin{aligned} & \sum_{|\alpha+\beta| \leq s-1, |\alpha| \leq |\beta|} \left\| \Gamma^\alpha \left(\frac{1 - n_1^2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \Gamma^\beta \left\{ \partial_\nu n_2 \partial_\mu \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \right\} \right\|_{L^1} \\ & + \sum_{|\alpha+\beta| \leq s-1, |\alpha| \leq |\beta|} \left\| \Gamma^\alpha \left(\frac{n_1 n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \Gamma^\beta \left\{ \partial_\nu n_1 \partial_\mu \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \right\} \right\|_{L^1} \\ & + C \sum_{|\alpha+\beta| \leq s-1, |\alpha| > |\beta|} \left\| \Gamma^\alpha \left(\frac{1 - n_1^2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \Gamma^\beta \left\{ \partial_\nu n_2 \partial_\mu \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \right\} \right\|_{L^1} \\ & + C \sum_{|\alpha+\beta| \leq s-1, |\alpha| > |\beta|} \left\| \Gamma^\alpha \left(\frac{n_1 n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \Gamma^\beta \left\{ \partial_\nu n_1 \partial_\mu \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \right\} \right\|_{L^1} \\ & \leq C \sum_{i=1}^2 \left\| \partial_\nu n_i \partial_\mu \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \right\|_{\Gamma, s-1, L^1} \\ & + C \left\{ \left\| \Gamma \left(\frac{1 - n_1^2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \right\|_{\Gamma, s-2, L^2} + \left\| \Gamma \left(\frac{n_1 n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \right\|_{\Gamma, s-2, L^2} \right\} \\ & \times \sum_{i=1}^2 \left\| \partial_\nu n_i \partial_\mu \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \right\|_{\Gamma, [\frac{s-1}{2}], L^2}. \end{aligned}$$

By (1.4) and Lemma 2.8, we hence conclude

$$\begin{aligned} & \left\{ \left\| \Gamma \left(\frac{1 - n_1^2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \right\|_{\Gamma, s-2, L^2} + \left\| \Gamma \left(\frac{n_1 n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \right\|_{\Gamma, s-2, L^2} \right\} \\ & \times \sum_{i=1}^2 \left\| \partial_\nu n_i \partial_\mu \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \right\|_{\Gamma, [\frac{s-1}{2}], L^2} \\ & \leq C \|(n_1, n_2)\|_{\Gamma, [\frac{s-1}{2}], L^\infty} \|(n_1, n_2)\|_{\Gamma, s-1, L^2} \\ & \quad \times \|(\partial n_1, \partial n_2)\|_{\Gamma, s-2, L^\infty}^2 \|(\partial n_1, \partial n_2)\|_{\Gamma, s, L^2} \\ & \leq C(M\epsilon)^5(1 + \tau)^{-1+3\delta}. \end{aligned}$$

On the other hand, using Lemma 2.9 and Lemma 2.10, one has

$$\begin{aligned}
& \sum_{i=1}^2 \left\| \partial_\nu n_i \partial_\mu \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \right\|_{\Gamma, s-1, L^1} \\
&= \sum_{i=1,2, |\alpha+\beta| \leq s-1, |\alpha| \geq |\beta|} \left\| \Gamma^\alpha \partial_\nu n_i \Gamma^\beta \partial_\mu \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \right\|_{L^1} \\
&\quad + \sum_{i=1,2, |\alpha+\beta| \leq s-1, |\alpha| < |\beta|} \left\| \Gamma^\alpha \partial_\nu n_i \Gamma^\beta \partial_\mu \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \right\|_{L^1} \\
&\leq C(1+\tau)^{-1} \left\{ \|(\partial n_1, \partial n_2)\|_{\Gamma, s-1, L^2} \left(\|(n_1, n_2)\|_{\Gamma, [\frac{s-1}{2}] + 2, L^\infty} \|(\partial n_1, \partial n_2)\|_{\Gamma, [\frac{s-1}{2}] + 1, L^2} \right) \right. \\
&\quad \left. + \|(\partial n_1, \partial n_2)\|_{\Gamma, [\frac{s-1}{2}], L^\infty} \left(\|(n_1, n_2)\|_{\Gamma, s, L^2} \|(\partial n_1, \partial n_2)\|_{\Gamma, s, L^2} \right) \right\} \\
&\leq C(M\epsilon)^3 (1+\tau)^{-1+3\delta}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
& \left\| \frac{n_1 (n_2^2 \partial_\mu n_1 \partial^\mu n_1 + n_1^2 \partial_\mu n_2 \partial^\mu n_2 - 2n_1 n_2 \partial_\mu n_1 \partial^\mu n_2)}{1 - n_1^2 - n_2^2} \right\|_{\Gamma, s-1, L^1} \\
&\leq C(M\epsilon)^3 [1 + (M\epsilon)^2] (1+\tau)^{-1+3\delta}.
\end{aligned} \tag{3.16}$$

Combining (3.14), (3.15) and (3.16), we arrive at

$$\|f_1(\tau, \cdot)\|_{\Gamma, s-1, L^1} \leq C(M\epsilon)^3 [1 + (M\epsilon)^2] (1+\tau)^{-1+3\delta}. \tag{3.17}$$

Inserting (3.17) into (3.9), one gets

$$\begin{aligned}
& \|n_1(t, \cdot)\|_{\Gamma, s-2, L^\infty} \leq C_\star [1 + (M\epsilon)^2] (1+t)^{-\frac{1}{2}} \\
& \quad \times \left\{ \|(n_{10}, n_{20})\|_{H^{s+2}} + \|(n_{11}, n_{21})\|_{H^{s+1}} + (M\epsilon)^3 \right\}
\end{aligned}$$

for some absolute positive constant C_\star . Repeating the above analysis, one can thus prove

$$\begin{aligned}
& \|n_2(t, \cdot)\|_{\Gamma, s-2, L^\infty} \leq C_\star [1 + (M\epsilon)^2] (1+t)^{-\frac{1}{2}} \\
& \quad \times \left\{ \|(n_{10}, n_{20})\|_{H^{s+2}} + \|(n_{11}, n_{21})\|_{H^{s+1}} + (M\epsilon)^3 \right\}.
\end{aligned}$$

One concludes that the third line in (3.2) is true provided that

$$\epsilon \leq \frac{1}{2M\sqrt{C_\star}}, \quad \|(n_{10}, n_{20})\|_{H^{s+2}} + \|(n_{11}, n_{21})\|_{H^{s+1}} \leq \frac{M\epsilon}{4C_\star}. \tag{3.18}$$

Estimates for $\|(n_1(t, \cdot), n_2(t, \cdot))\|_{\Gamma, s, L^2}$ in (3.2)

By Theorem 3.1 and using the similar proof as that for (3.9), one has

$$\begin{aligned} \|n_1(t, \cdot)\|_{\Gamma, s, L^2} &\leq C(1+t)^{\frac{1}{2}} (\|(n_{10}, n_{20})\|_{H^{s+2}} + \|(n_{11}, n_{21})\|_{H^{s+1}}) \\ &\quad + C(1+t)^{\frac{1}{2}} \int_0^t \left(\|f_1(\tau, \cdot)\|_{\Gamma, s, L^{\frac{4}{3}}, \chi_1} + (1+\tau)^{-\frac{1}{2}} \|f_1(\tau, \cdot)\|_{\Gamma, s, L^{1,2}, \chi_2} \right) d\tau \end{aligned} \quad (3.19)$$

and

$$\begin{aligned} \|n_2(t, \cdot)\|_{\Gamma, s, L^2} &\leq C(1+t)^{\frac{1}{2}} (\|(n_{10}, n_{20})\|_{H^{s+2}} + \|(n_{11}, n_{21})\|_{H^{s+1}}) \\ &\quad + C(1+t)^{\frac{1}{2}} \int_0^t \left(\|f_2(\tau, \cdot)\|_{\Gamma, s, L^{\frac{4}{3}}, \chi_1} + (1+\tau)^{-\frac{1}{2}} \|f_2(\tau, \cdot)\|_{\Gamma, s, L^{1,2}, \chi_2} \right) d\tau, \end{aligned} \quad (3.20)$$

where f_1 and f_2 are given in (3.7). Hence we need to estimate $\|f_j(\tau, \cdot)\|_{\Gamma, s, L^{\frac{4}{3}}, \chi_1}$ and $\|f_j(\tau, \cdot)\|_{\Gamma, s, L^{1,2}, \chi_2}$ for $j = 1, 2$.

They can be done as follows:

$$\begin{aligned} &\|f_1(\tau, \cdot)\|_{\Gamma, s, L^{\frac{4}{3}}, \chi_1} \\ &\leq C \left\| \frac{n_1}{1 - n_1^2 - n_2^2} \right\|_{\Gamma, [\frac{s}{2}], L^\infty} \left(\left\| \partial_\mu n_1 \partial^\mu n_1 + \partial_\mu n_2 \partial^\mu n_2 \right\|_{\Gamma, s, L^{\frac{4}{3}}, \chi_1} \right. \\ &\quad \left. + \left\| n_2^2 \partial_\mu n_1 \partial^\mu n_1 + n_1^2 \partial_\mu n_2 \partial^\mu n_2 - 2n_1 n_2 \partial_\mu n_1 \partial^\mu n_2 \right\|_{\Gamma, s, L^{\frac{4}{3}}, \chi_1} \right) \\ &\quad + C \left\| \frac{n_1}{1 - n_1^2 - n_2^2} \right\|_{\Gamma, s, L^2} \left(\left\| \partial_\mu n_1 \partial^\mu n_1 + \partial_\mu n_2 \partial^\mu n_2 \right\|_{\Gamma, [\frac{s}{2}], L^4, \chi_1} \right. \\ &\quad \left. + \left\| n_2^2 \partial_\mu n_1 \partial^\mu n_1 + n_1^2 \partial_\mu n_2 \partial^\mu n_2 - 2n_1 n_2 \partial_\mu n_1 \partial^\mu n_2 \right\|_{\Gamma, [\frac{s}{2}], L^4, \chi_1} \right) \\ &\quad + C \sum_{i=1}^2 \left\| \partial_\nu n_i \partial_\mu \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \right\|_{\Gamma, s, L^{\frac{4}{3}}, \chi_1} \\ &\quad + C \sum_{|\alpha+\beta| \leq s, |\alpha| > |\beta|} \left\| \Gamma^\alpha \left(\frac{1 - n_1^2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \Gamma^\beta \left\{ \partial_\nu n_2 \partial_\mu \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \right\} \right\|_{L^{\frac{4}{3}}, \chi_1} \\ &\quad + C \sum_{|\alpha+\beta| \leq s, |\alpha| > |\beta|} \left\| \Gamma^\alpha \left(\frac{n_1 n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \Gamma^\beta \left\{ \partial_\nu n_1 \partial_\mu \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \right\} \right\|_{L^{\frac{4}{3}}, \chi_1}. \end{aligned}$$

Next , we use Lemma 2.8 and Lemma 2.7, to obtain

$$\begin{aligned}
& \left\| \frac{n_1}{1 - n_1^2 - n_2^2} \right\|_{\Gamma, [\frac{s}{2}], L^\infty} \left(\left\| \partial_\mu n_1 \partial^\mu n_1 + \partial_\mu n_2 \partial^\mu n_2 \right\|_{\Gamma, s, L^{\frac{4}{3}}, \chi_1} \right. \\
& + \left\| n_2^2 \partial_\mu n_1 \partial^\mu n_1 + n_1^2 \partial_\mu n_2 \partial^\mu n_2 - 2n_1 n_2 \partial_\mu n_1 \partial^\mu n_2 \right\|_{\Gamma, s, L^{\frac{4}{3}}, \chi_1} \Big) \\
& + \left\| \frac{n_1}{1 - n_1^2 - n_2^2} \right\|_{\Gamma, s, L^2} \left(\left\| \partial_\mu n_1 \partial^\mu n_1 + \partial_\mu n_2 \partial^\mu n_2 \right\|_{\Gamma, [\frac{s}{2}], L^4, \chi_1} \right. \\
& + \left\| n_2^2 \partial_\mu n_1 \partial^\mu n_1 + n_1^2 \partial_\mu n_2 \partial^\mu n_2 - 2n_1 n_2 \partial_\mu n_1 \partial^\mu n_2 \right\|_{\Gamma, [\frac{s}{2}], L^4, \chi_1} \Big) \\
& \leq C \|(n_1, n_2)\|_{\Gamma, [\frac{s}{2}], L^\infty} \left[\|(\partial n_1, \partial n_2)\|_{\Gamma, s, L^2} \|(\partial n_1, \partial n_2)\|_{\Gamma, [\frac{s}{2}], L^4, \chi_1} \right. \\
& + \|(n_1, n_2)\|_{\Gamma, s, L^2} \|(\partial n_1, \partial n_2)\|_{\Gamma, [\frac{s}{2}], L^4, \chi_1} \|(n_1, n_2)\|_{\Gamma, [\frac{s}{2}] + 1, L^\infty}^2 \Big] \\
& + C(1 + \tau)^{-\frac{1}{2}} \|(n_1, n_2)\|_{\Gamma, s, L^2} \left(\left\| \partial_\mu n_1 \partial^\mu n_1 + \partial_\mu n_2 \partial^\mu n_2 \right\|_{\Gamma, [\frac{s}{2}] + 1, L^2} \right. \\
& + \left. \|(n_1, n_2)\|_{\Gamma, [\frac{s}{2}] + 1, L^\infty}^2 \sum_{i,j=1}^2 \left\| \partial^\mu n_i \partial_\mu n_j \right\|_{\Gamma, [\frac{s}{2}] + 1, L^2} \right) \\
& \leq C(1 + \tau)^{-\frac{1}{2}} \left\{ \|(n_1, n_2)\|_{\Gamma, [\frac{s}{2}], L^\infty} \|(\partial n_1, \partial n_2)\|_{\Gamma, s, L^2} \|(\partial n_1, \partial n_2)\|_{\Gamma, [\frac{s}{2}] + 1, L^2} \right. \\
& + (1 + \tau)^{-1} \|(n_1, n_2)\|_{\Gamma, s, L^2} \|(n_1, n_2)\|_{\Gamma, [\frac{s}{2}] + 2, L^\infty} \|(\partial n_1, \partial n_2)\|_{\Gamma, [\frac{s}{2}] + 1, L^2} \Big\} \\
& \leq C(M\epsilon)^3 (1 + \tau)^{-1+2\delta}.
\end{aligned}$$

In a similar manner, one deduces that

$$\begin{aligned}
& \sum_{i=1}^2 \left\| \partial_\nu n_i \partial_\mu \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \right\|_{\Gamma, s, L^{\frac{4}{3}}, \chi_1} \\
& \leq C \|(\partial n_1, \partial n_2)\|_{\Gamma, s, L^2} \left\| \frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right\|_{\Gamma, [\frac{s}{2}] + 1, L^4, \chi_1} \\
& + C \|(\partial n_1, \partial n_2)\|_{\Gamma, [\frac{s}{2}], L^4, \chi_1} \left\| \partial \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1 - n_1^2 - n_2^2}} \right) \right\|_{\Gamma, s, L^2} \\
& \leq C(1 + \tau)^{-\frac{1}{2}} \left[\|(\partial n_1, \partial n_2)\|_{\Gamma, s, L^2} + \|(\partial^2 n_1, \partial^2 n_2)\|_{\Gamma, s, L^2} \right] \|\partial n_1 \partial n_2\|_{\Gamma, [\frac{s}{2}] + 2, L^2} \\
& \leq C(M\epsilon)^3 (1 + \tau)^{-1+2\delta},
\end{aligned}$$

and that

$$\begin{aligned}
& \sum_{|\alpha+\beta|\leq s, |\alpha|>|\beta|} \left\| \Gamma^\alpha \left(\frac{1-n_1^2}{\sqrt{1-n_1^2-n_2^2}} \right) \Gamma^\beta \left\{ \partial_\nu n_2 \partial_\mu \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1-n_1^2-n_2^2}} \right) \right\} \right\|_{L^{\frac{4}{3}}, \chi_1} \\
& + C \sum_{|\alpha+\beta|\leq s, |\alpha|>|\beta|} \left\| \Gamma^\alpha \left(\frac{n_1 n_2}{\sqrt{1-n_1^2-n_2^2}} \right) \Gamma^\beta \left\{ \partial_\nu n_1 \partial_\mu \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1-n_1^2-n_2^2}} \right) \right\} \right\|_{L^{\frac{4}{3}}, \chi_1} \\
& \leq C \|(n_1, n_2)\|_{\Gamma, [\frac{s}{2}], L^\infty} \|(n_1, n_2)\|_{\Gamma, s, L^2} \left\| \partial_\nu n_2 \partial_\mu \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1-n_1^2-n_2^2}} \right) \right\|_{\Gamma, [\frac{s}{2}], L^4, \chi_1} \\
& \leq C(1+\tau)^{-\frac{1}{2}} \|(n_1, n_2)\|_{\Gamma, [\frac{s}{2}], L^\infty} \|(n_1, n_2)\|_{\Gamma, s, L^2} \\
& \quad \times \left\| \partial_\nu n_2 \partial_\mu \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1-n_1^2-n_2^2}} \right) \right\|_{\Gamma, [\frac{s}{2}]+1, L^2} \\
& \leq C(M\epsilon)^5 (1+\tau)^{-\frac{3}{2}+2\delta}.
\end{aligned}$$

Therefore, we have

$$\|f_1(\tau, \cdot)\|_{\Gamma, s, L^{\frac{4}{3}}, \chi_1} \leq C(M\epsilon)^3 (1+\tau)^{-1+2\delta}. \quad (3.21)$$

Next, by Hölder inequality and Theorem 2.5, one can proceed as follows:

$$\begin{aligned}
& \|f_1(\tau, \cdot)\|_{\Gamma, s, L^{1,2}, \chi_2} \\
& \leq C \left\| \frac{n_1}{1-n_1^2-n_2^2} \right\|_{\Gamma, [\frac{s}{2}], L^\infty} \left(\|\partial_\mu n_1 \partial^\mu n_1 + \partial_\mu n_2 \partial^\mu n_2\|_{\Gamma, s, L^{1,2}} \right. \\
& \quad \left. + \|n_2^2 \partial_\mu n_1 \partial^\mu n_1 + n_1^2 \partial_\mu n_2 \partial^\mu n_2 - 2n_1 n_2 \partial_\mu n_1 \partial^\mu n_2\|_{\Gamma, s, L^{1,2}} \right) \\
& \quad + C \left\| \frac{n_1}{1-n_1^2-n_2^2} \right\|_{\Gamma, s, L^2} \left(\|\partial_\mu n_1 \partial^\mu n_1 + \partial_\mu n_2 \partial^\mu n_2\|_{\Gamma, [\frac{s}{2}], L^{2,\infty}} \right. \\
& \quad \left. + \|n_2^2 \partial_\mu n_1 \partial^\mu n_1 + n_1^2 \partial_\mu n_2 \partial^\mu n_2 - 2n_1 n_2 \partial_\mu n_1 \partial^\mu n_2\|_{\Gamma, [\frac{s}{2}], L^{2,\infty}} \right) \\
& \quad + C \sum_{i=1}^2 \left\| \partial_\nu n_i \partial_\mu \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1-n_1^2-n_2^2}} \right) \right\|_{\Gamma, s, L^{1,2}} \\
& \quad + C \left\{ \left\| \Gamma \left(\frac{1-n_1^2}{\sqrt{1-n_1^2-n_2^2}} \right) \right\|_{\Gamma, s-1, L^2} + \left\| \Gamma \left(\frac{n_1 n_2}{\sqrt{1-n_1^2-n_2^2}} \right) \right\|_{\Gamma, s-1, L^2} \right\} \\
& \quad \times \sum_{i=1}^2 \left\| \partial_\nu n_i \partial_\mu \left(\frac{\partial^\mu n_1 \partial^\nu n_2 - \partial^\nu n_1 \partial^\mu n_2}{\sqrt{1-n_1^2-n_2^2}} \right) \right\|_{\Gamma, [\frac{s}{2}], L^{1,\infty}}.
\end{aligned}$$

Noting that

$$\begin{cases} \left\| \partial_\mu n_1 \partial^\mu n_1 + \partial_\mu n_2 \partial^\mu n_2 \right\|_{\Gamma, s, L^{1,2}} \\ \leq C \|(\partial n_1, \partial n_2)\|_{\Gamma, s, L^2} \|(\partial n_1, \partial n_2)\|_{\Gamma, [\frac{s}{2}], L^{2,\infty}} \\ \leq C \|(\partial n_1, \partial n_2)\|_{\Gamma, s, L^2} \|(\partial n_1, \partial n_2)\|_{\Gamma, [\frac{s}{2}]+1, L^2}, \\ \left\| \partial_\mu n_1 \partial^\mu n_1 + \partial_\mu n_2 \partial^\mu n_2 \right\|_{\Gamma, [\frac{s}{2}], L^{2,\infty}} \\ \leq C \left\| \partial_\mu n_1 \partial^\mu n_1 + \partial_\mu n_2 \partial^\mu n_2 \right\|_{\Gamma, [\frac{s}{2}]+1, L^2} \end{cases}$$

and using Theorem 2.5, one can further estimate

$$\|f_1(\tau, \cdot)\|_{\Gamma, s, L^{1,2}, \chi_2} \leq C(M\epsilon)^3(1+\tau)^{-\frac{1}{2}+2\delta}. \quad (3.22)$$

By inserting (3.21) into (3.19), one hence conclude

$$\begin{aligned} \|n_1(t, \cdot)\|_{\Gamma, s, L^2} &\leq C_\star \left[(1+t)^{\frac{1}{2}} (\|(n_{10}, n_{20})\|_{H^{s+2}} \right. \\ &\quad \left. + \|(n_{11}, n_{22})\|_{H^{s+1}}) + (M\epsilon)^3(1+t)^{\frac{1}{2}+2\delta} \right]. \end{aligned} \quad (3.23)$$

Repeating the above analysis, one has

$$\begin{aligned} \|n_2(t, \cdot)\|_{\Gamma, s, L^2} &\leq C_\star \left[(1+t)^{\frac{1}{2}} (\|(n_{10}, n_{20})\|_{H^{s+2}} \right. \\ &\quad \left. + \|(n_{11}, n_{22})\|_{H^{s+1}}) + (M\epsilon)^3(1+t)^{\frac{1}{2}+2\delta} \right]. \end{aligned} \quad (3.24)$$

By (3.23) and (3.24), we see that the second inequality in (3.2) is true provided that (3.18) is satisfied.

4 Energy Estimates

This section is devoted to estimating $\sum_{i=1}^2 \|\partial^i \mathbf{n}(t, \cdot)\|_{\Gamma, s, L^2}$ and proving the first inequality in (3.2). We begin with the following Lemma:

Lemma 4.1. *Let $n \geq 2$ and*

$$\text{supp } v, \text{supp } w \subset \{(t, x) : |x| \leq t + \rho\}.$$

Then for all $t \geq 0$:

$$\|v\partial w(t, \cdot)\|_{L^2} \leq C_\rho \|\nabla v(t, \cdot)\|_{L^2} \|\Gamma w(t, \cdot)\|_{L^\infty}.$$

Proof. By (2.5) in Proposition 2.2, we have

$$\begin{aligned} \|v\partial w(t, \cdot)\|_{L^2} &\leq C_\rho \left\| \frac{v\Gamma w(t, \cdot)}{\rho + |t - |x||} \right\|_{L^2} \\ &\leq C_\rho \left\| \frac{v}{\rho + |t - |x||} \right\|_{L^2} \|\Gamma w(t, \cdot)\|_{L^\infty} \\ &\leq C_\rho \|\nabla v\|_{L^2} \|\Gamma w(t, \cdot)\|_{L^\infty}. \end{aligned}$$

Here we used the following Hardy's inequality

$$\begin{aligned}
& \left\| \frac{v}{\rho + |t - |x||} \right\|_{L^2}^2 \leq C_\rho \int_{|\xi|=1} \int_0^{t+\rho} \frac{|v(r\xi)|^2}{(2\rho + t - r)^2} r^{n-1} dr dS \\
& = C_\rho \int_{|\xi|=1} \int_0^{t+\rho} |v(r\xi)|^2 r^{n-1} d \frac{1}{(2\rho + t - r)} dS \\
& = -C_\rho \int_{|\xi|=1} \int_0^{t+\rho} \frac{|v(r\xi)|^2}{2\rho + t - r} dr^{n-1} dS \\
& \quad - C_\rho \int_{|\xi|=1} \int_0^{t+\rho} \frac{2v(r\xi)v_r(r\xi)}{2\rho + t - r} r^{n-1} dr dS \\
& \leq -C_\rho \int_{|\xi|=1} \int_0^{t+\rho} \frac{2v(r\xi)v_r(r\xi)}{2\rho + t - r} r^{n-1} dr dS \\
& \leq C_\rho \|\nabla v\|_{L^2} \left\| \frac{v}{\rho + |t - |x||} \right\|_{L^2}.
\end{aligned}$$

□

Now let us rewrite the Fadeev model (1.3) as

$$\partial_\mu \partial^\mu \mathbf{n} + (\partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n}) \mathbf{n} + [\partial_\mu (\mathbf{n} \cdot [\partial^\mu \mathbf{n} \wedge \partial^\nu \mathbf{n}])] \partial_\nu \mathbf{n} \wedge \mathbf{n} = 0. \quad (4.1)$$

For $|\alpha| \leq s$ and $i = 0, 1$, similarly as in (3.8), one derives from (4.1) that

$$\square \partial^i \Gamma^\alpha \mathbf{n} = - \sum_{\beta \leq \alpha} C_{\alpha\beta} \partial^i \Gamma^\beta \left\{ (\partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n}) \mathbf{n} + [\partial_\mu (\mathbf{n} \cdot [\partial^\mu \mathbf{n} \wedge \partial^\nu \mathbf{n}])] \partial_\nu \mathbf{n} \wedge \mathbf{n} \right\}.$$

For $i = 0$ and 1 , taking the L^2 inner product of the above equations with $\partial_t \partial^i \Gamma^\alpha \mathbf{n}$ respectively and then adding them together, one has

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \sum_{i=0}^1 \sum_{|\alpha| \leq s} \|(\partial_t, \nabla) \partial^i \Gamma^\alpha \mathbf{n}\|_{L^2}^2 \\
& = - \sum_{i=0}^1 \sum_{|\alpha| \leq s} \sum_{\beta \leq \alpha} C_{\alpha\beta} \left\{ \int \partial_t \partial^i \Gamma^\alpha \mathbf{n} \cdot \partial^i \Gamma^\beta [(\partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n}) \mathbf{n}] dx \right. \\
& \quad \left. + \int \partial_t \partial^i \Gamma^\alpha \mathbf{n} \cdot \partial^i \Gamma^\beta [[\partial_\mu (\mathbf{n} \cdot [\partial^\mu \mathbf{n} \wedge \partial^\nu \mathbf{n}])] \partial_\nu \mathbf{n} \wedge \mathbf{n}] dx \right\}.
\end{aligned} \quad (4.2)$$

Let us estimate the first term on the right hand side of (4.2). A straightforward

calculation gives

$$\begin{aligned}
& - \sum_{i=0}^1 \sum_{|\alpha| \leq s} \sum_{\beta \leq \alpha} C_{\alpha\beta} \int \partial_t \partial^i \Gamma^\alpha \mathbf{n} \cdot \partial^i \Gamma^\beta [(\partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n}) \mathbf{n}] dx \\
& \leq C \sum_{i=0}^1 \sum_{|\alpha| \leq s} \sum_{\beta \leq \alpha} \|\partial_t \partial^i \Gamma^\alpha \mathbf{n} \cdot \mathbf{n}\|_{L^2} \|\partial^i \Gamma^\beta (\partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n})\|_{L^2} \\
& \quad + C \sum_{i=0}^1 \sum_{|\alpha| \leq s} \sum_{\gamma \leq \beta \leq \alpha, j \leq i, j+|\gamma| \geq 1} \|\partial_t \partial^i \Gamma^\alpha \mathbf{n}\|_{L^2} \\
& \quad \left\{ \sum_{j+|\gamma| \geq i-j+|\beta-\gamma|} \|\partial^j \Gamma^\gamma \mathbf{n}\|_{L^2} \|\partial^{i-j} \Gamma^{\beta-\gamma} (\partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n})\|_{L^\infty} \right. \\
& \quad \left. + \sum_{j+|\gamma| < i-j+|\beta-\gamma|} \|\partial^j \Gamma^\gamma \mathbf{n}\|_{L^\infty} \|\partial^{i-j} \Gamma^{\beta-\gamma} (\partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n})\|_{L^2} \right\}.
\end{aligned}$$

Noting the null structure of the nonlinearity and using (1.4), Lemma 2.9 and Lemma 2.10, we compute

$$\begin{aligned}
& \sum_{i=0}^1 \sum_{|\alpha| \leq s} \sum_{\gamma \leq \beta \leq \alpha, j \leq i, j+|\gamma| \geq 1} \|\partial_t \partial^i \Gamma^\alpha \mathbf{n}\|_{L^2} \tag{4.3} \\
& \left\{ \sum_{j+|\gamma| \geq i-j+|\beta-\gamma|} \|\partial^j \Gamma^\gamma \mathbf{n}\|_{L^2} \|\partial^{i-j} \Gamma^{\beta-\gamma} (\partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n})\|_{L^\infty} \right. \\
& \quad \left. + \sum_{j+|\gamma| < i-j+|\beta-\gamma|} \|\partial^j \Gamma^\gamma \mathbf{n}\|_{L^\infty} \|\partial^{i-j} \Gamma^{\beta-\gamma} (\partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n})\|_{L^2} \right\} \\
& \leq C(M\epsilon)^4 (1+t)^\delta (1+t)^{\frac{1}{2}+2\delta} (1+t)^{-2} + C(M\epsilon)^4 (1+t)^\delta (1+t)^{-1+\delta} \\
& \leq C(M\epsilon)^4 (1+t)^{-1+2\delta}.
\end{aligned}$$

On the other hand, by $\mathbf{n} \cdot \mathbf{n} = 1$ (which means $\mathbf{n} \cdot \mathbf{n}_t = 0$), one has

$$\begin{aligned}
& \mathbf{n} \cdot \partial^i \Gamma^\alpha \partial_t \mathbf{n} = - [\partial^i \Gamma^\alpha (\mathbf{n}_t \cdot \mathbf{n}) - \mathbf{n} \cdot \partial^i \Gamma^\alpha \partial_t \mathbf{n}] \\
& = - \sum_{0 \leq j \leq i, \beta \leq \alpha, j+|\beta| \geq 1} C_{j\beta} \partial^{i-j} \Gamma^{\alpha-\beta} \mathbf{n}_t \cdot \partial^j \Gamma^\beta \mathbf{n}.
\end{aligned} \tag{4.4}$$

Consequently, a similar argument as in (4.3) gives

$$\begin{aligned}
& \sum_{i=0}^1 \sum_{|\alpha| \leq s} \sum_{\beta \leq \alpha} \|\partial_t \partial^i \Gamma^\alpha \mathbf{n} \cdot \mathbf{n}\|_{L^2} \|\partial^i \Gamma^\beta (\partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n})\|_{L^2} \\
& \leq C(M\epsilon)^2 (1+t)^{-\frac{1}{2}+\delta} \sum_{i=0}^1 \sum_{|\alpha| \leq s} \sum_{\beta \leq \alpha, |\beta| \geq s-2} \|\partial^i \Gamma^{\alpha-\beta} \mathbf{n}_t \cdot \Gamma^\beta \mathbf{n}\|_{L^2} \\
& \quad + C(M\epsilon)^4 (1+t)^{-1+2\delta}.
\end{aligned}$$

Using Lemma 4.1, one can bound the right hand side of above equality by

$$C(M\epsilon)^4(1+t)^{-1+2\delta}.$$

Finally, one has

$$-\sum_{i=0}^1 \sum_{|\alpha| \leq s} \sum_{\beta \leq \alpha} C_{\alpha\beta} \int \partial_t \partial^i \Gamma^\alpha \mathbf{n} \cdot \partial^i \Gamma^\beta [(\partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n}) \mathbf{n}] dx \leq C(M\epsilon)^4(1+t)^{-1+2\delta}. \quad (4.5)$$

To estimate the right hand side of (4.2), it remains to bound

$$-\sum_{i=0}^1 \sum_{|\alpha| \leq s} \sum_{\beta \leq \alpha} C_{\alpha\beta} \left\{ \int \partial_t \partial^i \Gamma^\alpha \mathbf{n} \cdot \partial^i \Gamma^\beta \left[[\partial_\mu (\mathbf{n} \cdot [\partial^\mu \mathbf{n} \wedge \partial^\nu \mathbf{n}])] \partial_\nu \mathbf{n} \wedge \mathbf{n} \right] dx \right\}.$$

By a similar argument as (4.5), one can bound the above quantity by

$$-\sum_{i=0}^1 \int \partial_t \partial^i \Gamma^s \mathbf{n} \cdot (\partial_\nu \mathbf{n} \wedge \mathbf{n}) \partial_\mu (\mathbf{n} \cdot \partial^i \Gamma^s [\partial^\mu \mathbf{n} \wedge \partial^\nu \mathbf{n}]) dx + C(M\epsilon)^4(1+t)^{-1+2\delta},$$

which is equal to

$$\begin{aligned} & \sum_{i=0}^1 \int \partial_t \partial^i \Gamma^s \mathbf{n} \cdot (\nabla \mathbf{n} \wedge \mathbf{n}) \partial_t (\mathbf{n} \cdot \partial^i \Gamma^s [\partial_t \mathbf{n} \wedge \nabla \mathbf{n}]) dx \\ & + \sum_{i=0}^1 \int \partial_t \partial^i \Gamma^s \mathbf{n} \cdot (\partial_t \mathbf{n} \wedge \mathbf{n}) \nabla (\mathbf{n} \cdot \partial^i \Gamma^s [\nabla \mathbf{n} \wedge \partial_t \mathbf{n}]) dx \\ & - \sum_{i=0}^1 \int \partial_t \partial^i \Gamma^s \mathbf{n} \cdot (\partial_1 \mathbf{n} \wedge \mathbf{n}) \partial_2 (\mathbf{n} \cdot \partial^i \Gamma^s [\partial_2 \mathbf{n} \wedge \partial_1 \mathbf{n}]) dx \\ & - \sum_{i=0}^1 \int \partial_t \partial^i \Gamma^s \mathbf{n} \cdot (\partial_2 \mathbf{n} \wedge \mathbf{n}) \partial_1 (\mathbf{n} \cdot \partial^i \Gamma^s [\partial_1 \mathbf{n} \wedge \partial_2 \mathbf{n}]) dx \\ & + C(M\epsilon)^4(1+t)^{-1+2\delta}. \end{aligned}$$

Now let us rewrite the above quantity as

$$\begin{aligned} & \sum_{i=0}^1 \int ((\partial_t \partial^i \Gamma^s \mathbf{n} \wedge \nabla \mathbf{n}) \cdot \mathbf{n}) \partial_t (\mathbf{n} \cdot \partial^i \Gamma^s [\partial_t \mathbf{n} \wedge \nabla \mathbf{n}]) dx \\ & - \sum_{i=0}^1 \int \nabla ((\partial_t \mathbf{n} \wedge \partial_t \partial^i \Gamma^s \mathbf{n}) \cdot \mathbf{n}) (\mathbf{n} \cdot \partial^i \Gamma^s [\partial_t \mathbf{n} \wedge \nabla \mathbf{n}]) dx \\ & + \sum_{i=0}^1 \int \partial_2 ((\partial_1 \mathbf{n} \wedge \partial_t \partial^i \Gamma^s \mathbf{n}) \cdot \mathbf{n}) (\mathbf{n} \cdot \partial^i \Gamma^s [\partial_1 \mathbf{n} \wedge \partial_2 \mathbf{n}]) dx \\ & + \sum_{i=0}^1 \int \partial_1 ((\partial_t \partial^i \Gamma^s \mathbf{n} \wedge \partial_2 \mathbf{n}) \cdot \mathbf{n}) (\mathbf{n} \cdot \partial^i \Gamma^s [\partial_1 \mathbf{n} \wedge \partial_2 \mathbf{n}]) dx \\ & + C(M\epsilon)^4(1+t)^{-1+2\delta}. \end{aligned} \quad (4.6)$$

Similarly as in (4.5), one can estimate (4.6) by

$$\begin{aligned}
& \sum_{i=0}^1 \frac{1}{2} \frac{d}{dt} \left(\|\mathbf{n} \cdot \partial^i \Gamma^s [\partial_t \mathbf{n} \wedge \nabla \mathbf{n}]\|_{L^2}^2 + \|\mathbf{n} \cdot \partial^i \Gamma^s [\partial_1 \mathbf{n} \wedge \partial_2 \mathbf{n}]\|_{L^2}^2 \right) \\
& - 2 \int \left((\partial_t \mathbf{n} \wedge \partial^i \Gamma^s \nabla \mathbf{n}) \cdot \mathbf{n} \right) (\mathbf{n} \cdot \partial^i \Gamma^s [\partial_t \mathbf{n} \wedge \nabla \mathbf{n}]) dx + C(M\epsilon)^4 (1+t)^{-1+2\delta} \\
& \leq \sum_{i=0}^1 \frac{1}{2} \frac{d}{dt} \left(\|\mathbf{n} \cdot \partial^i \Gamma^s [\partial_1 \mathbf{n} \wedge \partial_2 \mathbf{n}]\|_{L^2}^2 + \|(\partial^i \Gamma^s \partial_t \mathbf{n} \wedge \nabla \mathbf{n}) \cdot \mathbf{n}\|_{L^2}^2 \right. \\
& \quad \left. - \|(\partial_t \mathbf{n} \wedge \partial^i \Gamma^s \nabla \mathbf{n}) \cdot \mathbf{n}\|_{L^2}^2 \right) + C(M\epsilon)^4 (1+t)^{-1+2\delta}.
\end{aligned} \tag{4.7}$$

Inserting (4.5) and (4.7) into (4.2), we finally arrive at

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \sum_{i=0}^1 \left(\sum_{|\alpha| \leq s} \|(\partial_t, \nabla) \partial^i \Gamma^\alpha \mathbf{n}\|_{L^2}^2 - \|\mathbf{n} \cdot \partial^i \Gamma^s [\partial_1 \mathbf{n} \wedge \partial_2 \mathbf{n}]\|_{L^2}^2 \right) \\
& - \|(\partial^i \Gamma^s \partial_t \mathbf{n} \wedge \nabla \mathbf{n}) \cdot \mathbf{n}\|_{L^2}^2 + \|(\partial_t \mathbf{n} \wedge \partial^i \Gamma^s \nabla \mathbf{n}) \cdot \mathbf{n}\|_{L^2}^2 \\
& \leq C(M\epsilon)^4 (1+t)^{-1+2\delta}.
\end{aligned} \tag{4.8}$$

This completes the proof of the first inequality in (3.2).

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